Extension of Strapdown Attitude Algorithm for High-Frequency Base Motion

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In this paper, we have expanded an application of the rotation vector concept to develop an efficient attitude algorithm for a strapdown system under a coning base motion. The proposed algorithm takes four samples of gyroscope data per update. A rotation vector error analysis for a coning motion has been employed to come up with an optimum four-sample algorithm. Also presented is an efficient way to implement the algorithm in a real-time environment. The algorithm is formulated in a recursive form using two time-rate executions and using only integer arithmetic. To compare the performance of the proposed algorithm with an already existing three-sample algorithm, a simulation has been performed by varying coning motion frequency and update rate. Results show that the four-sample algorithm outperforms the three-sample algorithm in case of high-frequency coning motion. The use of the four-sample algorithm is economical since it requires less computing time for an equal accuracy.

I. Introduction

A STRAPDOWN system for inertial navigation is characterized by the high dynamic range experienced by its sensors. Therefore, when we select an attitude algorithm for the system, we need to consider the effect of high-frequency base motion.

Commonly used attitude algorithms for a strapdown system are the Euler method, the direction cosine method, and the quaternion method.[1] Among them, the quaternion method is the most popular because of its advantages in nonsingularity, simplicity, and computation time.[2] Further improvement of the efficient quaternion method has been made possible by Bortz,[3] Miller,[2] and Jordan[4] employing the rotation vector concept. In fact, it is proven that the quaternion method with the rotation vector concept can effectively suppress the noncommutativity error, which is an important error source in attitude computation.

In this paper, we have expanded the rotation vector concept by taking four gyroscope data samples per attitude algorithm update, which provides an improved coning performance at high frequency and computational economy. A new strapdown attitude algorithm is derived, and an efficient way to implement the algorithm is also presented. To compare the performance of the new algorithm with the previously used algorithm based on three gyroscope data samples, a simulation is performed by varying coning motion frequency and update rate.

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II. Estimation of the Rotation Vector

The key operation in the attitude algorithm is to properly update the quaternion and rotation vector. The quaternion update $Q(T+h)$ is obtained by the following quaternion multiplication:

$$Q(T+h) = Q(T)^* q(h) \tag{1}$$

where $q(h)$ is an updating quaternion and is expressed by the usual quaternion relation,[1-5] viz.,

$$q(h) = \left(\cos\frac{\phi}{2}, \frac{\phi}{\phi}\ \sin\frac{\phi}{2}\right) \tag{2}$$

The updating quaternion $q(h)$ corresponds to an iteration that occurred during the time interval h. The rotation vector $\phi(t)$ in Eq. (2) can be expressed using a rate gyroscope output $\omega(t)$ as follows[3]:

$$\dot{\phi}(t) = \omega(t) + ½\phi(t) \times \omega(t) + 1/12\left[\phi(t) \times \left\{\phi(t) \times \omega(t)\right\}\right] \tag{3}$$

In Eq. (3) the triple-cross-product term is assumed to be small and can be neglected. The solution of Eq. (3) in the Taylor series is given by

$$\phi(T+h) = \phi(T) + h\dot{\phi}(T) + (h^2/2!)\ddot{\phi}(T) + \cdots \tag{4}$$

Now, we want to use four samples of gyroscope output to find the solution for $\phi(T+h)$. Using four sample points, the integration of gyroscope outputs during a small interval can be expressed in the following polynomial form:

$$\theta(t) = \int_0^t \omega(\tau)\,d\tau = at + bt^2 + ct^3 + dt^4, \qquad 0 \le t \le h \tag{5}$$

where a, b, c, and d are constant vectors. Repeated differentiation of Eq. (5) with respect to time gives the following relation at time $T(t=0)$:

$$\omega(T) = a, \quad \frac{d}{dt}\omega(T) = 2b, \quad \frac{d^2}{dt^2}\omega(T) = 6c$$

$$\frac{d^3}{dt^3}\omega(T) = 24d, \quad \text{and} \frac{d^4}{dt^4}\omega(T) = 0 \tag{6}$$

Differentiating Eq. (3) repeatedly with respect to time and applying Eq. (6) with initial condition $\phi(T)=0$ gives the following results:

$$\frac{d}{dt}\left[\phi(T)\right] = a \tag{7a}$$

$$\frac{d^2}{dt^2}\left[\phi(T)\right] = 2b \tag{7b}$$

$$\frac{d^3}{dt^3}\left[\phi(T)\right] = 6c + (a \times b) \tag{7c}$$

$$\frac{d^4}{dt^4}\left[\phi(T)\right] = 24d + 6(a \times c) \tag{7d}$$

$$\frac{d^5}{dt^5}\left[\phi(T)\right] = 36(a \times d) + 12(b \times c) \tag{7e}$$

$$\frac{d^6}{dt^6}\left[\phi(T)\right] = 120(b \times d) \tag{7f}$$

$$\frac{d^7}{dt^7}\left[\phi(T)\right] = 360(c \times d) \tag{7g}$$

$$\frac{d^8}{dt^8}\left[\phi(T)\right] = 0 \tag{7h}$$

where we intentionally ignored triple product terms because they are relatively small. Substituting Eqs. (7) into Eq. (4) brings Eq. (8) as follows:

$$\begin{aligned}\phi(T+h) &= ah + bh^2 + ch^3 + dh^4 + 1/6(a \times b)h^3 \\ &+ 1/4(a \times c)h^4 + 1/10(b \times c)h^5 + 3/10(a \times d)h^5 \\ &+ 1/6(b \times d)h^6 + 1/14(c \times d)h^7\end{aligned} \tag{8}$$

If we let the incremental gyroscope outputs at $t(=h/4$, $h/2$, $3h/4$, and $h)$ be θ_1, θ_2, θ_3, and θ_4, respectively, then $\theta = \theta_1 + \theta_2 + \theta_3 + \theta_4$. Putting this relationship into Eq. (5), we obtain four equations that yield the values of a, b, c, and d. If the values are substituted into Eq. (8), then we obtain a desired solution. That is,

$$\begin{aligned}\phi(T+h) &= \theta + K_1(\theta_1 \times \theta_2 + \theta_3 \times \theta_4) + K_2(\theta_1 \times \theta_3 + \theta_2 \times \theta_4) \\ &+ K_3(\theta_1 \times \theta_4) + K_4(\theta_2 \times \theta_3)\end{aligned} \tag{9}$$

where $K_1 = 736/945$, $K_2 = 334/945$, $K_3 = 526/945$, and $K_4 = 654/945$. However, the rotation vector estimated by Eq. (9) contains a large error. It needs to be further improved through an error analysis. It will be shown in the next section that the values of K_1, K_2, K_3, and K_4 can be adjusted through an optimization technique, resulting in a more exact value of the rotation vector.

III. Minimum Error Algorithm

Once we choose a base motion, we may be able to compute the corresponding exact quaternion. In this section, a coning motion is selected as base motion, and then the corresponding true quaternion is analytically computed for the chosen motion. On the other hand, the four-sample algorithm described in the previous section is applied to the motion to obtain an approximate quaternion. The difference of the two gives an error quaternion. After the error rotation vector is calculated from the error quaternion, the K values in Eq. (9) can be determined to minimize the magnitude of the error vector. In addition, an exact expression of the rotation vector error generated by the four-sample algorithm is derived in this section. It will be shown that the performance of the algorithm is improved by using the adjusted K values.

To analyze the performance of the attitude algorithm, a coning motion of cone half-angle ϕ and coning frequency ω is applied. The resulting body quaternion $Q(t)$ due to the coning motion is given by[2]

$$Q(t) = \begin{bmatrix} \cos(\phi/2) \\ 0 \\ \sin(\phi/2)\cos\omega t \\ \sin(\phi/2)\sin\omega t \end{bmatrix} \tag{10}$$

If $q(h)$ represents an incremental quaternion during an update time interval h under the coning motion, $Q(t+h)$ can be updated from $Q(t)$ using $q(h)$:

$$Q(t+h) = Q(t) * q(h) \tag{11}$$

or

$$q(h) = Q(t)^{-1} * Q(t+h) = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2\sin^2\left(\dfrac{\phi}{2}\right)\sin^2(\omega h/2) \\ -\sin^2\left(\dfrac{\phi}{2}\right)\sin(\omega h) \\ -\sin(\phi)\sin\left(\dfrac{\omega h}{2}\right)\sin\omega\left[t + (h/2)\right] \\ \sin(\phi)\sin\left(\dfrac{\omega h}{2}\right)\cos\omega\left[t + (h/2)\right] \end{bmatrix} \tag{12}$$

where $Q(t)$ and $q(h)$ represent the exact quaternions of the coning motion.

Now, we will show an approximate quaternion calculated by employing the proposed algorithm. The quaternion differential equation is generally written by[1-3]

$$\dot{Q}(t) = \frac{1}{2} Q(t) * \left[\omega(t)\right] \tag{13}$$

Therefore,

$$\left[\omega(t)\right] = 2Q^{-1}(t) * \dot{Q}(t) = \begin{bmatrix} -2\omega\sin^2(\phi/2) \\ -\omega\sin(\phi)\sin(\omega t) \\ \omega\sin(\phi)\cos(\omega t) \end{bmatrix} \tag{14}$$

where $[\omega(t)]$ represents gyroscope outputs in body coordinates. The output vector of rate integrating gyroscopes sensing body movements between time t and $(t+h)$ is

$$\theta(h)=\int_t^{t+h}\Big[\omega(\tau)\Big]\,\mathrm{d}\tau=\begin{bmatrix}-2\omega h\sin^2(\phi/2)\\ -2\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\sin\omega\Big[t+(h/2)\Big]\\ 2\sin(\phi)\sin\left(\frac{\omega h}{2}\right)\cos\omega\Big[t+(h/2)\Big]\end{bmatrix} \tag{15}$$

When four-sample sets per update period are taken, the values may be obtained from Eq. (15). They are given by

$$\theta_N(t)=\begin{bmatrix}-\frac{\omega h}{2}\sin^2\left(\frac{\phi}{2}\right)\\ \sin(\phi)\left[\cos\omega\left\{t+\frac{(Nh)}{4}\right\}-\cos\omega\left\{t+\frac{(N-1)h}{4}\right\}\right]\\ \sin(\phi)\left[\sin\omega\left\{t+\frac{(Nh)}{4}\right\}-\sin\omega\left\{t+\frac{(N-1)h}{4}\right\}\right]\end{bmatrix} \tag{16}$$

where N is 1, 2, 3, and 4 for θ_1, θ_2, θ_3, and θ_4, respectively. If these values are inserted into Eq. (9), it results after some manipulation, and assuming small ϕ,

$$\phi=[\phi_x,\ \phi_y,\ \phi_z]^T \tag{17}$$

where

$$\begin{aligned}\phi_x &= -2\omega h\sin^2(\phi/2)+\sin^2(\phi)\Big[(4K_1-2K_2+2K_4)\\ &\times\sin(\omega h/4)+(-2K_1+4K_2-K_3-K_4)\sin(2\omega h/4)\\ &+(-2K_2+2K_3)\sin(3\omega h/4)-K_3\sin(\omega h)\Big]\end{aligned}$$

$$\begin{aligned}\phi_y &= -2\sin(\phi)\sin(\omega h/2)\sin\omega\Big[\mathrm{t}+(h/2)\Big]\\ &+(\omega h/2)\sin^2(\phi/2)\sin(\phi)\Big[(K_1+K_2+K_3)\sin(\omega t)\\ &+(-2K_1-K_3+K_4)\sin\omega\Big[t+(h/4)\Big]\\ &+(2K_1-2K_2-2K_4)\sin\omega\Big[t+(h/2)\Big]\\ &+(-2K_1-K_3+K_4)\sin\omega\Big[t+(3h/4)\Big]\\ &+(K_1+K_2+K_3)\sin\omega(t+h)\Big]\end{aligned}$$

$$\begin{aligned}\phi_z &= 2\sin(\phi)\sin(\omega h/2)\cos\omega(t+h/2)\\ &+(\omega h/2)\sin^2(\phi/2)\sin(\phi)\Big[-(K_1+K_2+K_3)\cos(\omega t)\\ &-(-2K_1-K_3+K_4)\cos\omega\Big[t+(h/4)\Big]\\ &-(2K_1-2K_2-2K_4)\cos\omega\Big[t+(h/2)\Big]\\ &-(-2K_1-K_3+K_4)\cos\omega\Big[t+(3h/4)\Big]\\ &-(K_1+K_2+K_3)\cos\omega(t+h)\Big]\end{aligned}$$

It should be pointed out that, to maintain the accuracy of this approximation, the computer iteration rate must be high enough so that ϕ remains small under maximum vehicle rotation rate condition.

The quaternion estimate $\hat{q}(h)$ due to the rotation vector of Eq. (17) is then given by

$$\hat{q}(h)=\begin{bmatrix}\cos(\phi/2)\\ (\phi_x/\phi)\sin(\phi/2)\\ (\phi_y/\phi)\sin(\phi/2)\\ (\phi_z/\phi)\sin(\phi/2)\end{bmatrix}\equiv\begin{bmatrix}C\\ \phi_x S\\ \phi_y S\\ \phi_z S\end{bmatrix} \tag{18}$$

Now $\tilde{q}(h)$ is defined as follows:

$$\tilde{q}(h)=q(h)^*\hat{q}^{-1}(h)$$

or

$$\begin{bmatrix}\tilde{q}_0\\ \tilde{q}_1\\ \tilde{q}_2\\ \tilde{q}_3\end{bmatrix}=\begin{bmatrix}q_0C-S(-q_1\phi_x-q_2\phi_y-q_3\phi_z)\\ q_1C-S(q_0\phi_x-q_3\phi_y+q_2\phi_z)\\ q_2C-S(q_3\phi_x+q_0\phi_y-q_1\phi_z)\\ q_3C-S(-q_2\phi_x+q_1\phi_y+q_0\phi_z)\end{bmatrix} \tag{19}$$

where $q(h)$ is the true quaternion shown in Eq. (12). If $\hat{q}(h)$ does not contain an error, $\tilde{q}(h)$ becomes a unity. Therefore, the terms representing $\tilde{q}_1$, $\tilde{q}_2$, and $\tilde{q}_3$ in Eq. (19) imply an estimation error.

An inspection of $q(h)$ and ϕ in Eqs. (12) and (17) shows that q_2, q_3, ϕ_y, and ϕ_z are all periodic with frequency equal to the coning frequency. Equation (19) shows that $\tilde{q}_2$ and $\tilde{q}_3$ are also periodic, and they contribute a reciprocating error. We are only interested in nonperiodic errors in $\tilde{q}(h)$, because they are the ones causing drift rate error during the quaternion update of $Q(t+h)$. Such terms appear in $\tilde{q}_0$ and $\tilde{q}_1$.

Since ϕ is assumed to be small, we may put $C=1$ and $S=½$ in Eq. (18). The nonperiodic term $\tilde{q}_0$ in Eq. (19) is q_0C, which is 1 as seen in Eq. (12). The nonperiodic term of $\tilde{q}_1$ is $(q_1C-Sq_0\phi_x)$, which can be reduced to the following form:

$$\tilde{q}_1(h)=q_1-\frac{1}{2}\phi_x \tag{20}$$

The quaternion error $\tilde{q}_1(h)$ may be written in a quaternion standard form such as

$$\tilde{q}_1(h)=\phi_x/\phi_\epsilon\sin(\phi_\epsilon/2)=\sin(\phi_\epsilon/2)$$

where (ϕ_x/ϕ_ϵ) equals 1 since the error is concentrated in the x axis. The variable ϕ_ϵ is the magnitude of the error rotation. If ϕ_ϵ is small, approximately $\phi_\epsilon=2\tilde{q}_1$. Substituting the result into Eq. (20) gives

$$\phi_\epsilon=2q_1-\phi_x \tag{21}$$

Substituting the values of q_1 and ϕ_x obtained from Eqs. (12) and (17) into Eq. (21), we obtain

$$\begin{aligned}\phi_\epsilon &= 2\sin^2(\phi/2)\Big\{\omega h-\sin(\omega h)\Big\}-\sin^2(\phi)\\ &\times\Big[(4K_1-2K_2+2K_4)\sin(\omega h/4)\\ &+(-2K_1+4K_2-K_3-K_4)\sin(\omega h/2)\\ &+(-2K_2+2K_3)\sin(3\omega h/4)-K_3\sin(\omega h)\Big]\end{aligned} \tag{22}$$

If the sine terms in Eq. (22) are expanded by power series, Eq.

(22) can be written as

$$\phi_\epsilon = \phi^2\Big[(\omega h)^3(-1/384)\{12K_1+24K_2+18K_3+6K_4-32\}$$
$$+ (\omega h)^5(1/122,880)\{60K_1+360K_2+570K_3$$
$$+30K_4-512\} + (\omega h)^7(-1/82,575,360)\{252K_1$$
$$+ 3864K_2+12,138K_3+126K_4-8192\}$$
$$+ (\omega h)^9(1/95,126,814,720)\{1020K_1+37,320K_2$$
$$+ 223,290K_3+510K_4-131,072\} + O\{(\omega h)^{11}\}\Big] \tag{23}$$

Now, we can solve for K_1, K_2, K_3, and K_4 setting the preceding error equations to zero. However, the preceding four equations are dependent; the K are not uniquely determined. To uniquely determine the K, we adopt the concept of "distance" between the cross products in Eq. (9). The cross products can be ordered according to the distance between the first vector and the second as shown by

Distance 1:

$$\theta_1\times\theta_2,\qquad \theta_2\times\theta_3,\qquad \theta_3\times\theta_4$$

Distance 2:

$$\theta_1\times\theta_3,\qquad \theta_2\times\theta_4$$

Distance 3:

$$\theta_1\times\theta_4$$

It is then found that cross products with equal "distance" behave exactly the same for coning inputs. Thus, individual weighting is not important; only the total weight of terms with the same distance is relevant. The rotation vector estimation, Eq. (9), can then be rewritten as

$$\phi(T+h) = \theta + K_5\Big[a_1(\theta_1\times\theta_2) + a_2(\theta_2\times\theta_3) + a_3(\theta_3\times\theta_4)\Big]$$
$$+ K_6\Big[b_1(\theta_1\times\theta_3) + b_2(\theta_2\times\theta_4)\Big] + K_3(\theta_1\times\theta_4) \tag{24}$$

with the constraints

$$a_1 + a_2 + a_3 = 1,\qquad b_1 + b_2 = 1 \tag{25}$$

The error equation of the rotation vector, ϕ_ϵ, can now be written employing this newly found rotation vector estimation as follows:

$$\phi_\epsilon = \phi^2\Big[(\omega h)^3(-1/384)\{6K_5+12K_6+18K_3-32\}$$
$$+ (\omega h)^5(1/122,880)\{30K_5+180K_6+570K_3-512\}$$
$$+ (\omega h)^7(-1/82,575,360)\{126K_5+1932K_6+12,138K_3$$
$$-8192\} + (\omega h)^9(1/95,126,814,720)\{510K_5$$
$$+ 18,660K_6+223,290K_3-131,072\}\Big] \tag{26}$$

If we set the coefficients of $(\omega h)^3$, $(\omega h)^5$, and $(\omega h)^7$ to zero, we get $K_5=214/105$, $K_6=92/105$, and $K_3=54/105$, which provide a unique estimation of the rotation vector. When these values of K_5, K_6, and K_3 are substituted into Eq. (26), we obtain a minimum error value of the rotation vector; that is,

$$\phi_\epsilon = \frac{\phi^2(\omega h)^9}{82,575,360} + \text{high-order terms} \tag{27}$$

The corresponding algorithm to estimate rotation vector from gyroscope output is given by

$$\phi(T+h) = \theta + 54/105(\theta_1\times\theta_4)$$
$$+ 92/105\Big[b_1(\theta_1\times\theta_3) + b_2(\theta_2\times\theta_4)\Big]$$
$$+ 214/105\Big[a_1(\theta_1\times\theta_2)+a_2(\theta_2\times\theta_3)+a_3(\theta_3\times\theta_4)\Big] \tag{28}$$

where a_1, a_2, a_3, b_1, and b_2 can be arbitrarily determined as long as they satisfy the constraints defined in Eq. (25).

This algorithm, compared to two- and three-sample algorithms, produces much less estimation error. For the comparison, coning error rates of various algorithms are tabulated in Table 1. The table shows that the four-sample algorithm gives much less error than the other two.

Another useful comparison is the relative coning error, which is the ratio of the error rate to coning rate.[6] Results of the relative coning error in a wide frequency range between various algorithms are summarized in Fig. 1. In the figure, relative coning errors are plotted against coning frequency multiplied by sampling time (dt). The figure shows that the four-sample algorithm outperforms the two- and three-sample algorithm throughout the frequency range considered.

IV. Real-Time Implementation of the Algorithm

It has been demonstrated in the previous section that the four-sample algorithm, when it is used for a strapdown inertial system with high-frequency base motion, can vastly improve the accuracy of computing attitudes. Considering a real-time implementation and coding of the algorithm in a digital com-

Table 1 Coning error rate of various algorithms

Algorithm	Estimation of rotation vector	Coning error, rad/s
2-Sample	$\phi = \theta + 2/3(\theta_1\times\theta_2)$	$\phi^2\omega^5h^4/960$
3-Sample	$\phi = \theta + 9/20(\theta_1\times\theta_3) + 27/40\theta_2 \times(\theta_3-\theta_1)$	$\phi^2\omega^7h^6/204,120$
4-Sample	Eq. (28)	$\phi^2\omega^9h^8/82,575,360$

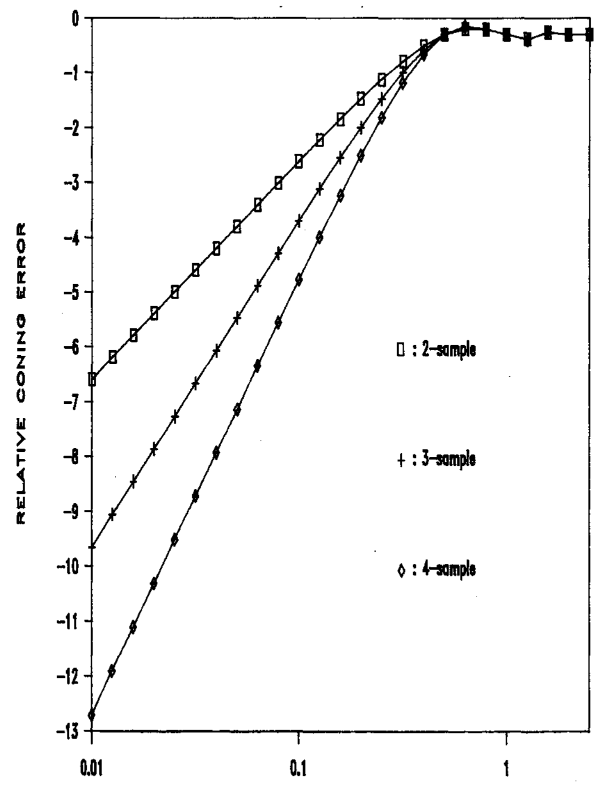

Fig. 1 Coning response (log-log plot).

puter, we need to solve two problems; floating-point arithmetic and recursive calculation. The four-sample algorithm proposed in this paper as well as the two- or three-sample algorithms previously published, in their present forms, require floating-point arithmetic. In addition, for recursive calculation, the rotation vector estimation and quaternion update have to be carried out in each update interval however small the interval is.

In this paper, we present a general scheme that can ease implementing the attitude algorithms in a recursive form using high- and low-rate executions and using only integer arithmetic. Before a real-time algorithm is formulated, some notations on update interval are defined and illustrated in Fig. 2. In the figure, three time intervals are defined; the smallest interval dt represents a sampling time, the minor interval consists of four dt in the case of a four-sample algorithm, and the major interval consists of a number of minor intervals. It is desirable that the rotation vector be updated in every minor interval, whereas the quaternion is updated only once in every major interval. In that way, the time to compute quaternion update can be reduced while the rotation vector is updated at a high rate. To make it possible, an algorithm has to be obtained to directly update the rotation vector instead of using a quaternion. At the same time, it is desirable that the algorithm use integer arithmetic rather than floating arithmetic.

To obtain a recursive form of rotation vector update, a modification to Eq. (28) is made as follows[7]:

$$\begin{aligned}\boldsymbol{\phi}(t) = \boldsymbol{\theta} + \tfrac{1}{2}\boldsymbol{\phi}\times\boldsymbol{\theta} + 92/105\Big[b_1(\boldsymbol{\theta}_1\times\boldsymbol{\theta}_3) + b_2(\boldsymbol{\theta}_2\times\boldsymbol{\theta}_4)\Big]\\ + 214/105\Big[a_1(\boldsymbol{\theta}_1\times\boldsymbol{\theta}_2) + a_2(\boldsymbol{\theta}_2\times\boldsymbol{\theta}_3) + a_3(\boldsymbol{\theta}_3\times\boldsymbol{\theta}_4)\Big]\\ + 54/105(\boldsymbol{\theta}_1\times\boldsymbol{\theta}_4)\end{aligned} \tag{29}$$

Now we separate Eq. (29) into two parts—high and low rates—as shown in Eqs. (30) and (31).

High rate:

$$\boldsymbol{\theta} = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 + \boldsymbol{\theta}_3 + \boldsymbol{\theta}_4 \tag{30a}$$

$$\begin{aligned}\boldsymbol{R}_1 := \boldsymbol{R}_1 + x_1(\boldsymbol{\phi}\times\boldsymbol{\theta}) + x_2\Big\{(\boldsymbol{\theta}_1\times\boldsymbol{\theta}_3) + (\boldsymbol{\theta}_2\times\boldsymbol{\theta}_4)\Big\}\\ + x_3(\boldsymbol{\theta}_1\times\boldsymbol{\theta}_4) + x_4(\boldsymbol{\theta}_2\times\boldsymbol{\theta}_3)\end{aligned} \tag{30b}$$

$$\begin{aligned}\boldsymbol{R}_2 := \boldsymbol{R}_2 + y_1\Big\{(\boldsymbol{\theta}_1\times\boldsymbol{\theta}_2) + (\boldsymbol{\theta}_3\times\boldsymbol{\theta}_4)\Big\} + y_2\Big\{(\boldsymbol{\theta}_1\times\boldsymbol{\theta}_3) + (\boldsymbol{\theta}_2\times\boldsymbol{\theta}_4)\Big\}\\ + y_3(\boldsymbol{\theta}_1\times\boldsymbol{\theta}_4)\end{aligned} \tag{30c}$$

$$\boldsymbol{\phi} := \boldsymbol{\phi} + \boldsymbol{\theta} \tag{30d}$$

Low rate:

$$\boldsymbol{\theta} := \boldsymbol{\phi} + z_1(\boldsymbol{R}_1 + z_2\boldsymbol{R}_2) \tag{31}$$

where x_1, x_2, x_3, x_4, y_1, y_2, and y_3 are small integers. The high-rate computations [Eqs. (30)] are performed for every minor interval, and the low-rate computation [Eq. (31)] is performed for every major interval. In the preceding algorithm, $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, and $\boldsymbol{\phi}$ are reset to zero at the end of every major interval. Substituting Eqs. (30) into Eq. (31) and comparing the result with Eq. (29), the following relations have to hold in order for the two equations to be equal:

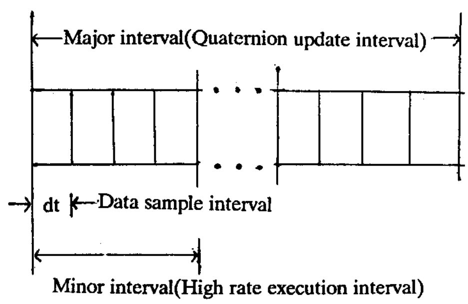

Fig. 2 Interval definitions.

$$z_1x_1 = 1/2 \tag{32a}$$

$$z_1z_2y_1 = (214/105)a_1 \tag{32b}$$

$$z_1x_4 = (214/105)a_2 \tag{32c}$$

$$z_1x_2 + z_1z_2y_2 = 46/105 \tag{32d}$$

$$z_1x_3 + z_1z_2y_3 = 54/105 \tag{32e}$$

If we solve Eq. (32) with the smallest integer coefficients, we get $z_1 = 1/6$, $z_2 = 16/35$, $x_1 = 3$, $x_2 = x_3 = x_4 = 4$, $y_1 = 9$, $y_2 = -3$, and $y_3 = -2$. When these values are substituted into Eqs. (30) and (31), we obtain a desired real-time algorithm. That is,

High rate:

$$\boldsymbol{\theta} = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 + \boldsymbol{\theta}_3 + \boldsymbol{\theta}_4 \tag{33a}$$

$$\boldsymbol{R}_1 := \boldsymbol{R}_1 + \Big\{3\boldsymbol{\phi} + 4(\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2)\Big\}\times\boldsymbol{\theta} \tag{33b}$$

$$\boldsymbol{R}_2 := \boldsymbol{R}_2 + \boldsymbol{\theta}_1\times(9\boldsymbol{\theta}_2 - 3\boldsymbol{\theta}_3 - \boldsymbol{\theta}_4) + (9\boldsymbol{\theta}_3 - 3\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)\times\boldsymbol{\theta}_4 \tag{33c}$$

$$\boldsymbol{\phi} := \boldsymbol{\phi} + \boldsymbol{\theta} \tag{33d}$$

Low rate:

$$\boldsymbol{\phi} := \boldsymbol{\phi} + 1/6\Big[\boldsymbol{R}_1 + (16/35)\boldsymbol{R}_2\Big] \tag{34}$$

The rotation vector $\boldsymbol{\phi}$ calculated at every major interval is used to obtain the corresponding quaternion. In other words, the quaternion is updated from $\boldsymbol{\phi}$ in contrast to its own update explained in Sec. III. Therefore, employing the algorithm shown in Eqs. (33) and (34), the quaternion update does not have to be done in each update interval. Thus, we can choose the quaternion update interval as we like.

Quaternions obtained employing this newly developed algorithm are identical to those of employing the algorithm proposed in the previous section. This real-time implementation technique can equally be applied to two- or three-sample algorithms.

V. Simulation

To verify the improvement of the proposed algorithm over the existing algorithms, simulations for different algorithms using three and four samples are done by varying the coning frequency and update rate. In all of the simulations, the real-time algorithm developed in Sec. IV is employed. The coning frequency of the selected base motion varies in wide range, and the variation of update rate ranges from 0.06 to 0.01 s. Since the performance degradation caused by quantization effects can be considered the same for all of the algorithms regardless of the number of samples taken in an interval,[2] we can ignore the effect of quantization in comparing attitude algorithms.

Simulation results are summarized in Figs. 3 and 4. The relative coning error of the rotation vector for three- and four-sample algorithms against coning frequency multiplied by sampling time (dt) are plotted in Fig. 3, and rotation vector errors against update rate are plotted in Fig. 4. It can be seen from Fig. 3 that the simulation results are close to the theoretical computation shown in Fig. 1, and the four-sample algorithm shows consistently better performance than the three-sample algorithm in all of the ranges of coning frequency considered. The maximum accuracy improvement of the four-sample algorithm over the three-sample algorithm is in an order of 3. It can be seen from Fig. 4 that the rotation vector

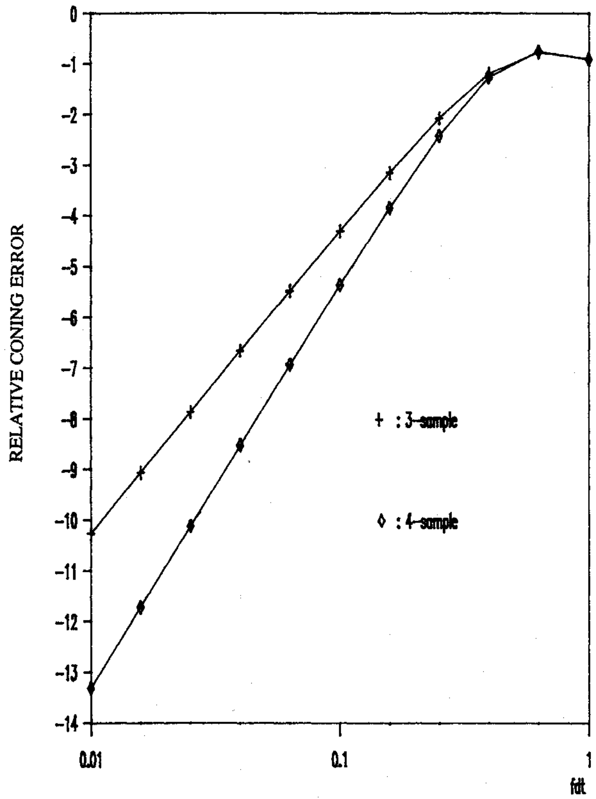

Fig. 3 Relative coning error vs coning frequency.

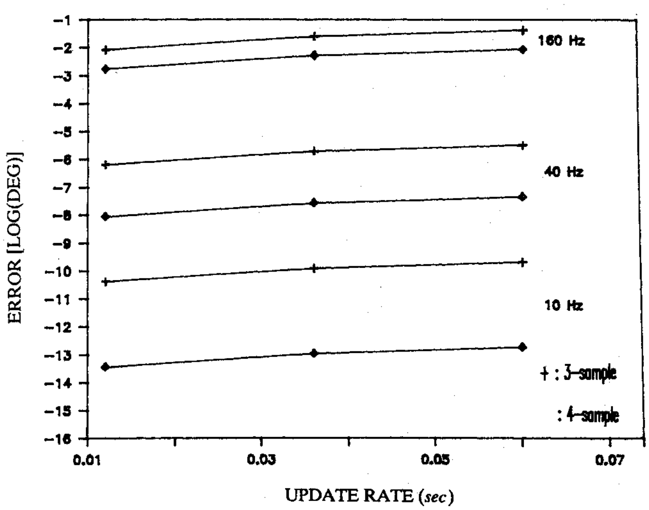

Fig. 4 Rotation vector error vs update rate.

errors of the four-sample algorithm are about the same order as those with the three-sample algorithm when the coning frequency is about 2–3 times greater. Also, it can be seen from the figure that the performance of the algorithm is not significantly affected by the update rate change.

Calculation of the four-sample algorithm during a sampling time interval requires 9 multiplications and 12 additions, whereas the three-sample algorithm requires 6 multiplications and 10 additions. That is, computer loading of the four-sample algorithm is approximately 30% greater than that of the three-sample algorithm, and the accuracy improvement is an order of 2. Therefore, the four-sample algorithm is more economical than the two- or three-sample algorithm. In any case, the four-sample algorithm outperforms the three-sample algorithm in the case of the coning motion.

VI. Conclusion

The four-sample algorithm presented in this paper deals with the problem of high-frequency base motion in applications where an accurate attitude calculation is required. The mechanization can be characterized in a number of aspects. The algorithm uses the rotation vector and minimizes its error. That is accomplished by employing a "distance" concept between sampled data. The error improvement of the four-sample algorithm over previously existing two- or three-sample mechanizations is achieved in any frequency range of base motion. The four-sample algorithm has a relative coning error that is eighth order in frequency. This should be compared with fourth- and sixth-order relative errors for the second- and third-sample algorithm, respectively. Higher-order algorithms using a larger number of samples can also be devised.

Another important aspect of the proposed four-sample algorithm is that the mechanization is divided into two time-scale computations. The high-rate algorithm that updates the rotation vector can be performed entirely in integer arithmetic. The time-consuming quaternion update is performed at a lower rate, resulting in much savings in the real-time implementation of the attitude integration. Dual rate schemes are also applicable to two- or three-sample algorithms. However, the four-sample case is advantageous with respect to the three-sample case since binary rate ratios are much more desirable in digital computers.

Simulations varying coning frequency and update rate demonstrate that the four-sample algorithm outperforms the three-sample algorithm. The simulation results coincide with the theoretical expectations. The improvement is maintained regardless of the quaternion update rate.

In conclusion, the four-sample attiude integration algorithm is recommended for use where high-accuracy attitude is required under conditions of high-frequency base motion.

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References

[1]Nurse, R. J., Prohaska, J., and Riegsecker, D., *A New Baseline for the Inertial Navigation Strapdown Simulation*, Vol. 1, Charles Stark Draper Lab., Cambridge, MA, R-1136, 1978.

[2]Miller, R. B., "A New Strapdown Attitude Algorithm," *Journal of Guidance, Control, and Dynamics*, Vol. 6, No. 4, 1983, pp. 287–291.

[3]Bortz, J. E., "A New Mathematical Formulation for Strapdown Inertial Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 7, No. 1, 1971, pp. 61–66.

[4]Jordan, J. W., "An Accurate Strapdown Direction Cosine Algorithm," NASA TND-5384, Sept. 1969.

[5]Wilcox, J. C., "A New Algorithm for Strapped-Down Inertial Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 3, No. 5, 1967, pp. 796–802.

[6]McKern, R., "A Study of Transformation Algorithms for Use in Digital Computer," M.S. Thesis, Massachusetts Institute of Technology, Cambridge, MA, Jan. 1968.

[7]Savage, P. G., "Strapdown System Algorithms," AGARD-LS-133, 1984.